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Exam Two MTH–221 , Summer 2022

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Score = $\frac{\cancel{46}\ 45}{46}$ Excellent ++

QUESTION 1. (16 points)

(i) Let A be a 4×4 matrix such that $C_A(\alpha) = (\alpha - 2)(\alpha - 3)^2(\alpha - b)$, where $b \in R$, and $|A| = 72$. Then $Trace(A) =$

+1

(a)15 (b) 9 (c) 11 (d) -9 (e) 12

(ii) Let $T : R^2 \to R^2$ be an R-homomorphism (linear transformation) with eigenvalues 2, 0 such that $E_2 = span\{(2,1)\}$ and $E_0 = span\{(-4,-1)\}$. Then the set of all points in R^2 where $T(a,b) = -2(a,b)$ is

(a) $Span\{(-2,-1)\}$ (b) $\{(0,0)\}$ (c) $\{(-2,-1)\}$ (d) $Span\{(-2,0)\}$

(iii) Let $A = \begin{bmatrix} 1 & -2 & -4 \\ -1 & a & b \\ -2 & 4 & c \end{bmatrix}$. Then A^{-1} exists in one of the following cases.

(a) $a = 2,\ b = 54$, and $c = 19$ (b) $a = 13,\ b = 0$, and $c = 8$ (c) $a = 0,\ b = 34$, and $c = 39$
(d) $a = 14,\ b = 43$, and $c = 8$

(iv) Let $A = \begin{bmatrix} 1 & a & 5 \\ b & c & d \\ 2 & e & 4 \end{bmatrix}$ such that $|A| = 4$. Let $B = \begin{bmatrix} 1 & a & 5 \\ b & c+2 & d \\ 2 & e & 4 \end{bmatrix}$. Then $|B| =$ (hint: use the second row to find |A| and |B|. Then stare)

(a) 6 (b) 2 (c) 16 (d) -8

(v) Let $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$. Then the eigenvalues of A

(a) 2, 4 (b) 1 , 5 (c) 2, 3 (d) -2, -4

(vi) Given A is a 3×3 matrix with eigenvalues 1, 2, 3. Then $|A + 6A^{-1}| =$

(a) $\frac{37}{6}$ (b) 7 (c) 175 (d) $\frac{7}{6}$

(vii) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ such that $|A| = 2022$. Let D be the solution set of the system of linear equations $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1 + a_3 \\ b_1 + b_3 \\ c_1 + c_3 \end{bmatrix}$. Then

(a) D is infinite, but not a subspace (b) $D = \{\}$ (c) $D = Span\{(1,0,1)\}$ (d) $D = \{(1,0,1)\}$

(viii) Let $A = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$ such that $|A| = 4$. Given $B = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 3c_1 & 3c_2 & 3c_3 \\ 2b_1 & 2b_2 & 2b_3 \end{bmatrix}$. Then $|B| =$

(a) 24 (b) -12 (c) 12 (d) -24

QUESTION 2. Let $T : R^3 \to R^3$ be an R-homomorphism (i.e., Linear Transformation) such that $T(a,b,c) = (2a,\ -a+b-c,\ -a-b+c)$.

(i) (4 points) Find all eigenvalues of T.

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$C_T(a) = |aI_3 - T| = \left| \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \right| = \begin{vmatrix} a-2 & 0 & 0 \\ 1 & a-1 & 1 \\ 1 & 1 & a-1 \end{vmatrix}$$

$$(a-2)(-1)^{1+1} \begin{vmatrix} a-1 & 1 \\ 1 & a-1 \end{vmatrix} = (a-2)\left[(a-1)(a-1)-1\right] = (a-2)(a^2-2a+\cancel{1}-\cancel{1})$$

$$= (a-2)(a^2-2a) = a(a-2)(a-2)$$

Eigenvalues: $a=0$ (repeated once)
$a=2$ (twice)

(ii) (4 points) For each eigenvalue a of T, find E_a and write it as span.

$a=0$
$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow{-\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right] \xrightarrow[-R_1+R_3\to R_3]{-R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} a=0 \\ b=c \\ 0=0 \\ c \text{ free} \end{array}$$

$\{(0,c,c) \mid c \in \mathbb{R}\}$

$E_0 = \text{span}\{(0,1,1)\}$

$\dim(E_0) = 1$

$a=2$
$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-R_2+R_3\to R_3} \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} 0=0 \\ a+b+c=0 \Rightarrow a=-b-c \\ 0=0 \end{array}$$

b, c free

$\{(-b-c, b, c) \mid b, c \in \mathbb{R}\}$

$E_2 = \text{span}\{(-1,1,0), (-1,0,1)\}$

$\dim(E_2) = 2$

QUESTION 3. Let $T: R^4 \to R^4$ be an R-homomorphism (i.e., Linear Transformation) such that $T(a,b,c,d) = (a+b+c+d,\ -a-b-c,\ -2a-2b-2c-2d,\ -3a-3b-3c-3d)$. Let M be the standard matrix presentation of T.

(i) (4 points) Find Rank(M). Then find dim(Range(T))

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 0 \\ -2 & -2 & -2 & -2 \\ -3 & -3 & -3 & -3 \end{bmatrix} \xrightarrow[\substack{2R_1+R_3\to R_3 \\ 3R_1+R_4\to R_4}]{R_1+R_2\to R_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{Rank}(M) = \dim(\text{Range}(T)) = 2$

(ii) (4 points) by staring at (i), find a basis for $Col(M)$. Then find a basis for the Range(T). 1^{st} and 4^{th} columns

a basis for $Col(M) = \{(1,-1,-2,-3), (1,0,-2,-3)\}$

a basis for the $Range(T)$ = basis for $Col(M) = \{(1,-1,-2,-3), (1,0,-2,-3)\}$

QUESTION 4. (i) (4 points) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix}$ Find A^{-1} if possible

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 \\ -1 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{-R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right] \xrightarrow{-R_3+R_1\to R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

(ii) (2 points) Let A as in (i), find the solution set to the system of linear equations given by $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}$

Multiply by A^{-1} from the left

$$\underbrace{\cancel{A^{-1}A}}_{I_3}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -1 & -\frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 6 \end{bmatrix}$$

Solution Set $S = \{(-6, 2, 6)\}$

QUESTION 5. (i) (4 points) Find a matrix A, 2×2, such that $\left(\begin{bmatrix}0 & 1\\ \frac{1}{3} & -1\end{bmatrix} A^{-1}\right)^{-1} + A = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$

$(AB)^{-1} = B^{-1}A^{-1}$: $\left(\begin{bmatrix}0 & 1\\ 1/3 & -1\end{bmatrix} A^{-1}\right)^{-1} = A \frac{1}{-1/3}\begin{bmatrix}-1 & -1\\ -1/3 & 0\end{bmatrix}$

$A\begin{bmatrix}3 & 3\\ 1 & 0\end{bmatrix} + A = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$

$A\left(I_2\begin{bmatrix}3 & 3\\ 1 & 0\end{bmatrix} + I_2\right) = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$

$A\left(\begin{bmatrix}3 & 3\\ 1 & 0\end{bmatrix} + \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right) = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$

$A\left(\underbrace{\begin{bmatrix}4 & 3\\ 1 & 1\end{bmatrix}}_{B}\right) = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}$

$A\underbrace{BB^{-1}}_{I_2} = \begin{bmatrix}1 & 1\\ 1 & 2\end{bmatrix}\overbrace{\begin{bmatrix}1 & -3\\ -1 & 4\end{bmatrix}}^{B^{-1}}$

$A = \begin{bmatrix}0 & 1\\ -1 & 5\end{bmatrix}$ ✓ Good

$B^{-1} = \frac{1}{4-3}\begin{bmatrix}1 & -3\\ -1 & 4\end{bmatrix} = \begin{bmatrix}1 & -3\\ -1 & 4\end{bmatrix}$

(ii) (4 points) Let A be a 4×4 matrix. Given

$$A \xrightarrow{2R_3} \underset{2|A|}{B} \xrightarrow{R_4 \leftrightarrow R_2} \underset{-2|A|}{C} \xrightarrow{-4R_3 + R_1 \to R_1} D = \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 2 & 1\\ -2 & 0 & 0 & 2\\ -1 & -1 & -1 & 3\end{bmatrix}$$

$$\xrightarrow[R_1+R_4 \to R_4]{2R_1+R_3 \to R_3} \cancel{F} = \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 0 & 2 & 1\\ 0 & 2 & 2 & 4\\ 0 & 0 & 0 & 4\end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \underset{2|A|}{F} = \begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 2 & 2 & 4\\ 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 4\end{bmatrix}$$

$|F| = (1)(2)(2)(4) = 16$ ✓

a. Find $|A|$ $= \frac{1}{2}|F| = 8$ ✓

b. Find $|0.5C|$ $= 0.5^4|C| = (0.5^4)(-2)|A| = (0.5^4)(-2)(8) = -1$ ✓